NASA Contractor Report 4108

Investigation, Development, and Application of Optimal Output Feedback Theory

Volume IV: Measures of Eigenvalue/ Eigenvector Sensitivity to System Parameters and Unmodeled Dynamics

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Prepared for
Langley Research Center
under Contract NAS1-15759

National Aeronautics
and Space Administration

Scientific and Technical
Information Division

1987

FOREWORD

The work described in this report was performed by Information & Control Systems, Incorporated under Contract Number NAS1-15759 for the National Aeronautics and Space Administration, Langley Research Center, Hampton, Virginia. The work was sponsored by the Guidance and Control Division, Aircraft Guidance and Control Branch at Langley Research Center. Mr. R. M. Hueschen was the NASA Technical Representative monitoring this contract. Dr. N. Halyo directed the technical effort at ICS.

ABSTRACT

In this report, some measures of eigenvalue and eigenvector sensitivity applicable to both continuous and discrete linear systems are developed and investigated. An infinite series representation is developed for the eigenvalues and eigenvectors of a system. The coefficients of the series are coupled, but can be obtained recursively using a nonlinear coupled vector difference equation.

A new sensitivity measure is developed by considering the effects of unmodeled dynamics. It is shown that the sensitivity is high when any unmodeled eigenvalue is near a modeled eigenvalue. Using a simple example where the sensor dynamics have been neglected, it is shown that high feedback gains produce high eigenvalue/eigenvector sensitivity. The smallest singular value of the return difference is shown not to reflect eigenvalue sensitivity since it increases with the feedback gains.

Using an upper bound obtained from the infinite series, a procedure to evaluate whether the sensitivity to parameter variations is within given acceptable bounds is developed and demonstrated by an example.

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I. INTRODUCTION

Most complex systems of interest contain uncertainties in at least some parameters. Beyond this aspect, system parameters vary over the range of operating points in which the system is supposed to perform its basic objective. Finally, the system design model usually neglects some dynamic effects in order to simplify the design process and to accommodate nonlinear effects and avoid determining some subsystem characteristics. Therefore, in the design of the feedback control law, one of the most important considerations is the closed-loop system's robustness.

In this study, by system robustness we shall mean that the basic system properties or characteristics are sufficiently insensitive to the various system parameter variations and modeling errors to perform the basic system objective. The feedback controller modifies the system matrix (A) to achieve a variety of system characteristics such as transient response, sensor and plant noise suppression, feedforward bandwidth, stability, etc. However, it is necessary that these characteristics be performed in an acceptable manner despite any unmodeled dynamics and other system parameter variations.

To achieve these closed-loop system characteristics, the designer of the feedback control law has a variety of methods at his disposal ranging from stochastic optimization methods such as full-state feedback LQG, output feedback, pole placement, classical frequency techniques, etc. The use of the stochastic output feedback and feedforward design techniques developed by the author [1] - [5] have proved to be suitable in many cases.

When a feedback design is being considered, it is necessary to evaluate the robustness of the resulting closed-loop system. Since the feedback design produces a closed-loop system matrix, and this matrix is completely determined by its eigenvalues and eigenvectors, it seems reasonable to assume that the desired system characteristics are implicitly contained in the closed-loop eigenstructure. Therefore, to evaluate the sensitivity of the system characteristics designed, it may be sufficient to consider the sensitivity of the system's eigenvalues and eigenvectors. In this investigation, we will develop some measures of eigenvalue and eigenvector sensitivity to evaluate a given feedback design from the standpoint of robustness and try to establish some trends.

Eigenvalue sensitivity has received some attention in the literature [6] - [18]. The majority of the work on eigenstructure sensitivity deals with the 1^{st} derivative of the eigenvalues and sometimes the eigenvectors with respect to a parameter. In some cases, the 1^{st} derivative of the state trajectory is investigated. In other cases, the sensitivity of the type-1 property is studied.

More recently, a frequency domain approach to robustness has been introduced [19] - [24]. This approach stems from a desire to extend classical SISO sensitivity measures to MIMO systems. The measure of robustness used is the smallest singular value of the return difference matrix as a function of frequency.

In this report, measures of system sensitivity based on the closed-loop eigenvalues and eigenvectors are developed. A new measure of sensitivity is introduced by investigating the effect of unmodeled dynamics. This measure is applied to an example where the dynamics of a sensor are neglected in the model, as is often the case. It is shown that as the system gains are increased, the actual system eigenvalue sensitivity increases and may even lead to instability. Therefore, this measure of sensitivity requires that the feedback gains be maintained at lower levels if possible.

The fact that high feedback gains cause significant robustness problems, is usually first encountered when a high-gain design based on a linear model is simulated on a noisy and fully nonlinear simulation. In most complex physical systems, unmodeled dynamics occur due to nonlinear effects and the complexity which would be required to model all structural modes, electronic harmonics, etc. Thus the unmodeled dynamics are essentially a fact of life for the control designer.

The sensitivity to unmodeled dynamics places a limit to how high the loop gains can realistically be set.

The smallest singular value of the return difference, however, generally increases with increasing feedback gains. This implies that the higher the feedback gains, the higher the system robustness! This trend of the singular value based sensitivity approach is puzzling and leads to the conclusion that the smallest singular value of the return difference is more a measure of stability margin and plant noise suppression properties rather than system sensitivity or robustness.

Then, infinite series representations of the system eigenvalues and eigenvectors are developed. It is shown that the coefficients of the power series for the eigenvalues and eigenvectors can be recursively obtained. From this series representation, an upper bound on the variation in the eigenvalues and eigenvectors is obtained and applied to evaluate the sensitivity of an example.

II. AN INFINITE SERIES REPRESENTATION OF EIGENVALUES AND EIGENVECTORS

Some of the most important characteristics of a linear system, whether continuous or digital, are contained in the eigenvalues and eigenvectors of that system. For example, the stability of the system is completely determined by the location of the eigenvalues; similarly, the damping ratios, the zeroes and poles, the modal behavior, etc. are determined by the system eigenvalues and eigenvectors.

The linear systems which will be considered in this study are of the form

$$\dot{x} = A\,x + B\,u \qquad , \tag{1}$$

for continuous time systems, and

$$x_{k+1} = \phi\,x_k + \Gamma\,u_k \qquad , \tag{2}$$

for discrete time systems. In both the continuous and discrete system cases, the state x is considered to be a n-vector and the control u a r-vector. Thus, the system matrices A and ϕ both have the dimensions $n \times n$, while B and Γ are dimensioned $n \times r$.

The system eigenvalues and eigenvectors are defined as those of the matrices A or ϕ according to whether the continuous system (1) or the discrete system (2) is under investigation. Thus, in both cases, we are interested in the eigenvalues and eigenvectors of a square matrix: In the following, we will derive an infinite series for the eigenvalues of the square matrix A; the results, of course, are equally applicable to the discrete system matrix ϕ.

In most systems of interest, the system matrix, A, varies with the operating point; so that the system matrix $A(p)$ is a function of some parameter, say p. In this study, we are interested in determining the variation in the eigenvalues and eigenvectors of the matrix $A(p)$ as the parameter p varies.

A. Power Series for Eigenvalues and Eigenvectors

Now suppose that we have an infinite series representation of the system matrix $A(p)$

$$A(p) = \sum_{k=0}^{\infty} A_k \, p^k \qquad , \tag{3}$$

Also suppose that $\lambda_i(p)$ and $x_i(p)$ form an eigenvalue–eigenvector pair for $A(p)$; i.e.,

$$A(p) \; x_i(p) = \lambda_i(p) \; x_i(p) \qquad , \qquad 1 \le i \le n \qquad . \tag{4}$$

Note that $p = 0$ corresponds to the case where $A(0) = A_o$ is the system matrix for the reference operating point. Now, suppose that the eigenvalue $\lambda_i(p)$ and the eigenvector $x_i(p)$ have infinite series representations of the form

$$\lambda_i(p) = \sum_{k=0}^{\infty} \lambda_{ik} \, p^k \qquad , \qquad 1 \le i \le n \qquad , \tag{5}$$

$$x_i(p) = \sum_{k=0}^{\infty} x_{ik} \, p^k \qquad , \qquad 1 \le i \le n \qquad , \tag{6}$$

Let us denote the smallest of the radii of convergence for the power series in (3), (5) and (6) by d. So that all three power series converge absolutely within $[-d,\, d]$. Unless stated otherwise, we will consider p to belong to the interval $[-d,\, d]$.

In order to understand and evaluate the changes in the system matrix $A(p)$ as the parameter, p, varies within $[-d,\, d]$, it is of interest to find expressions for the coefficients of the infinite series, i.e., λ_{ik} and x_{ik}. Rewriting (4), we obtain

$$\left[\sum_{k=0}^{\infty} A_k \, p^k\right]\left[\sum_{j=0}^{\infty} x_{ij} \, p^j\right] = \left[\sum_{k=0}^{\infty} \lambda_{ik} \, p^k\right]\left[\sum_{j=0}^{\infty} x_{ij} \, p^j\right] \tag{7}$$

Using well-known properties of power series [1], we obtain that within $[-d,\, d]$,

$$\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} A_j \; x_{ik-j}\right] p^k = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} \lambda_{ij} \; x_{ik-j}\right] p^k \qquad , \qquad 1 \le i \le n \tag{8}$$

$$\sum_{j=0}^{k}(\lambda_{ij}\ I - A_j)\ x_{ik-j} = 0 \quad , \quad k \geq 0 \quad , \quad 1 \leq i \leq n \tag{9}$$

From (9), it is seen that the coefficients of the power series satisfy a homogeneous difference equation of non-finite order. The case of $k = 0$, can be seen to be a restatement of (4) when $p = 0$; i.e.,

$$(\lambda_{io}\ I - A_o)\ x_{io} = 0 \quad , \quad 1 \leq i \leq n \quad , \tag{10}$$

which states that the coefficients λ_{io} and x_{io} form an eigenvalue–eigenvector pair for the matrix $A(0) = A_o$. This is already clear from (5) and (6); however, the higher order coefficients are the ones which contain information about the variations in the eigenstructure of $A(p)$ as p varies within $[-d,\ d]$. For example, when $k = 1$, we have

$$(\lambda_{io}\ I - A_o)\ x_{i1} = (A_1 - \lambda_{i1}\ I)\ x_{io} \tag{11}$$

Since (11) contains $n+1$ scalar unknowns (i.e., x_{i1} and λ_{i1}), but only contains n equations, it may appear that a solution cannot be obtained. However, since the eigenvector, $x_i(p)$, is only determined up to a constant factor, the coefficients x_{ij} maintain one degree of freedom. On the other hand, the coefficients, λ_{ik}, are unique and can be determined from the set of equations in (9).

To simplify the derivation, consider the case where $A(0)$ has distinct eigenvalues. Minor variations are required to treat the case where $A(0)$ has eigenvalues of multiplicity greater than 1. Let e_i be the i^{th} column of the $n \times n$ identity matrix, and let X_o be the non-singular matrix whose columns are the eigenvectors of $A(0)$; i.e.,

$$X_o\ e_i = x_{io} \quad , \quad 1 \leq i \leq n \quad . \tag{12}$$

It follows that

$$A_o = X_o \, \Lambda_o \, X_o^{-1} \qquad , \tag{13}$$

where Λ_o is a diagonal matrix containing the eigenvalues, $\{\lambda_{io}, 1 \leq i \leq n\}$, of $A(0)$. Let c_{ik} be defined by

$$X_o \, c_{ik} = x_{ik} \qquad , \qquad c_{ik} = X_o^{-1} \, x_{ik} \qquad , \qquad k \geq 0 \qquad , \qquad 1 \leq i \leq n \qquad . \tag{14}$$

Now (9) can be restated as

$$\sum_{j=0}^{k} (\lambda_{ij} \, I - \Lambda_j) \, c_{ik-j} = 0 \qquad , \qquad k \geq 0 \qquad , \qquad 1 \leq i \leq n \qquad , \tag{15}$$

$$\Lambda_j = X_o^{-1} \, A_j \, X_o \qquad , \qquad j \geq 0 \qquad , \tag{16}$$

where Λ_j is not necessarily diagonal when $j > 0$, but can be easily computed from (16).

To show that we can recursively compute λ_{ik} and c_{ik}, assume that $\{\lambda_{ij}, c_{ij}, 0 \leq j \leq k-1\}$ are known and note that

$$c_{io} = e_i \qquad , \qquad 1 \leq i \leq n \qquad , \tag{17}$$

$$e_i^T (\lambda_{io} \, I - \Lambda_o) \, c_{ik} = 0 \qquad , \qquad 1 \leq i \leq n \qquad , \qquad k \geq 0 \tag{18}$$

where (17) follows from (14); now pre-multiplying (15) by e_i^T and using (17) and (18), it is seen that*

$$\lambda_{ik} = e_i^T \, \Lambda_k \, e_i + \sum_{j=1}^{k-1} e_i^T (\Lambda_j - \lambda_{ij} \, I) \, c_{ik-j} \qquad , \qquad k \geq 0 \tag{19}$$

*Throughout this work, we use the convention that $\sum_{j=1}^{k} a_j = 0$, whenever $k < i$.

$$\lambda_{ik} = \sum_{j=1}^{k} e_i^T \Lambda_j \, c_{ik-j} - \sum_{j=1}^{k-1} (e_i^T \, c_{ik-j}) \, \lambda_{ij} \quad , \quad k \geq 1 \quad . \tag{20}$$

Note that λ_{ik} does not depend on c_{ik} as a result of (18) and can be computed using either of the equivalent expressions in (19) or (20). Now, to obtain the expression for c_{ik}, rewrite (15) in the form

$$(\lambda_{io} \, I - \Lambda_o) \, c_{ik} = \sum_{j=1}^{k} (\Lambda_j - \lambda_{ij} \, I) \, c_{ik-j} \quad , \quad k \geq 1 \tag{21}$$

As noted in (18), the diagonal matrix $(\lambda_{io} \, I - \Lambda_o)$ has a zero as its i^{th} diagonal element; so that the left-hand-side (LHS) of (21) is independent of the i^{th} element of c_{ik}; i.e., $e_i^T \, c_{ik}$. Therefore, the i^{th} element of c_{ik} cannot be obtained from (21). However, the remaining elements of c_{ik} can, in fact, be obtained from (21) since all the remaining elements on the diagonal of $(\lambda_{io} \, I - \Lambda_o)$ are non-zero.

Let $(\lambda_{io} \, I - \Lambda_o)^{\#}$ represent the following pseudo-inverse

$$(\lambda_{io} \, I - \Lambda_o)^{\#} = \sum_{j=1, j \neq i}^{k} (\lambda_{io} - \lambda_{jo})^{-1} \, e_j \, e_j^T \quad , \quad 1 \leq i \leq n \quad . \tag{22}$$

It follows that

$$c_{ik} = (\lambda_{io} \, I - \Lambda_o)^{\#} \sum_{j=1}^{K} (\Lambda_j - \lambda_{ij} \, I) \, c_{ik-j} + c_{iki} \, e_i \; , \; k \geq 1 \; , \; 1 \leq i \leq n \; , \tag{23}$$

where c_{iki} is the i^{th} component of c_{ik} which is not determined from (21).

Since λ_{ik} is independent of c_{ik}, it can be computed first and substituted into (23) to obtain the components of c_{ik}, except for the i^{th} component which will be shown to be arbitrary, presently. It follows by induction that all the coefficients λ_{ik} and c_{ik} can be obtained recursively using (20) and (23).

Alternatively, suppose that we have the sequences $\{\lambda_{ik}, c_{ik}, k \geq 0\}$ which satisfy (19) and (23), where c_{iki} is an arbitrary complex number for each k, and where $c_{io} = e_i$.

Pre-multiplying (23) by $(\lambda_{io}\ I - \Lambda_o)$ and using (19) for the i^{th} row results in (21) which implies (19). If the power series (5) and (6) converge for the above sequences, it follows that $\lambda_i(p)$ and $x_i(p)$ form an eigenvalue–eigenvector pair for the matrix $A(p)$ within the mutual convergence region of (3), (5) and (6).

From the preceding, it is clear that the i^{th} component of c_{ik} can be selected arbitrarily for each k. As mentioned earlier, since the eigenvectors of a matrix are only determined up to scalar factor, it is not surprising that the coefficients of the expansion for the eigenvectors have a degree of freedom. One approach is to select c_{iki} so as to simplify the formulae and computation of λ_{ik} and c_{ik}. An obvious choice would be

$$c_{iki} = e_i^T\ c_{ik} = 0 \quad , \quad k \geq 1 \quad , \quad 1 \leq i \leq n \quad . \tag{24}$$

This selection would result in selecting the eigenvector $x_i(p)$ which satisfies

$$e_i^T\ X_o^{-1}\, x_i(p) = 1 \quad , \quad p \in [-d,\, d] \quad , \quad 1 \leq i \leq n \quad . \tag{25}$$

Thus, unless the eigenvector $x_i(p)$ is orthogonal to its corresponding row-eigenvector at $p = 0$, i.e., $e_i^T\ X_o^{-1}$, the selection (24) is valid.

With the selection of (24), or alternatively (25), the coefficients for the infinite series of the eigenvalues and eigenvectors become

$$\lambda_{ik} = \sum_{j=1}^{k} e_i^T\ \Lambda_j\ c_{ik-1} \quad , \quad k \geq 1 \quad , \quad 1 \leq i \leq n \quad , \tag{26}$$

$$c_{ik} = (\lambda_{io}\ I - \Lambda_o)^{\#} \sum_{j=1}^{k} (\Lambda_j - \lambda_{ij}\ I)\ c_{ik-j} \ , \ k \geq 1 \ , \ 1 \leq i \leq n \ . \tag{27}$$

$$c_{io} = e_i \quad , \quad e_i^T\ \Lambda_o\ e_i = \lambda_{io} \quad , \quad 1 \leq i \leq n \tag{28}$$

Finally, it is important to note that while the eigenvectors and their coefficients are not unique, the eigenvalues and their coefficients are unique. In other words, no matter what selection of eigenvectors is made, the numerical value obtained for λ_{ik} will be the same. This is very significant as it implies that the sensitivity of the eigenvalues of a linear system is independent of the particular representation of the system, which is a desirable property; i.e., sensitivity is a property of the system itself rather than its particular representation.

B. Expressions for Some Coefficients

Thus, we have shown that when the system matrix $A(p)$ can be expressed in the form of (3), as the parameter, p, varies, the system eigenvalues and eigenvectors also vary according to (5) and (6), where the coefficients are given by (26), (27) and (28). The change is the eigenvalues $\lambda_i(p)$ as p varies within the convergence region can be evaluated by analyzing the coefficients, λ_{ik}. Thus, using (26) and (27),

$$\lambda_{i1} = e_i^T \ \Lambda_1 \ c_{io} = e_i^T \ \Lambda_1 \ e_i \qquad , \qquad 1 \leq i \leq n \tag{29}$$

$$\lambda_{i1} = e_i^T \ X_o^{-1} \ A_1 \ X_o \ e_i \qquad , \qquad 1 \leq i \leq n \tag{30}$$

It is interesting to note that λ_{i1} depends only on A_1 but not on higher order terms such as A_2, A_3, etc. Having computed λ_{i1}, we can now obtain c_{i1} and λ_{i2}.

$$c_{i1} = (\lambda_{io} \ I - \Lambda_o)^{\#} \ (\Lambda_1 - \lambda_{i1} \ I) \ e_i \qquad , \qquad 1 \leq i \leq n \quad . \tag{31}$$

Using (31) in (19), we obtain

$$\lambda_{i2} = e_i^T \ \Lambda_2 \ e_i + e_i^T (\Lambda_1 - \lambda_{i1} \ I) \ (\lambda_{io} \ I - \Lambda_o)^{\#} \ (\Lambda_1 - \lambda_{i1} \ I) \ e_i \ , \ 1 \leq i \leq n \tag{32}$$

$$\begin{aligned} c_{i2} = (\lambda_{io} \ I - \Lambda_o)^{\#} \big[& (\Lambda_1 - \lambda_{i1} \ I) \ (\lambda_{io} \ I - \Lambda_o)^{\#} \ (\Lambda_1 - \lambda_{i1} \ I) \\ & + (\Lambda_2 - \lambda_{i2} \ I) \big] \ e_i \ , \ 1 \leq i \leq n \ . \end{aligned} \tag{33}$$

It can be shown that λ_{i1} and λ_{i2} remain unchanged if the eigenvectors are selected differently than (27), by using the more general expressions in (23) and (19). The resulting expressions for λ_{i1} and λ_{i2} can be reduced to (29) and (32) after some manipulation which will not be included here.

From the preceding, it is clear that the coefficients for the power series for the system eigenvalues and eigenvectors can be easily computed using the recursive algorithms derived.

C. Upper Bounds on Eigenvalue Variations

The variations in the eigenvalues $\lambda_i(p)$ as p varies can be closely approximated in the vicinity of $p = 0$ by using the first one or two terms of the power series developed and neglecting the remainder. While this provides the more significant information and will be discussed later in more detail, it is also of interest to have a upper bound on the eigenvalue variations, in particular, when large variations in the parameter, p, are being considered. Usually, the large variations are of interest in order to establish characteristics such as the stability margin rather than the tracking control performance characteristics.

In the following, we will develop an upper bound on the change in each eigenvalue using the infinite series obtained for the eigenvalues and eigenvectors. To simplify the derivation, we will consider the case

$$A(p) = A_o + A_1\ p = X_o\ (\Lambda_o + \Lambda_1\ p)\ X_o^{-1} \qquad , \tag{34}$$

$$A_k = \Lambda_k = 0 \qquad , \qquad k \geq 2 \qquad . \tag{35}$$

Rewriting (26) and (27) for each i, we have

$$\lambda_{ik} = e_i^T\ \Lambda_1\ c_{ik-1} \qquad , \qquad k \geq 1 \tag{36}$$

$$c_{ik} = (\lambda_{io}\, I - \Lambda_o)^{\#}\left[\Lambda_1\, c_{ik-1} - \sum_{j=1}^{k} \lambda_{ij}\, c_{ik-j}\right] \quad , \qquad k \geq 1 \tag{37}$$

From (36) and (37), it follows that

$$|\lambda_{ik}| \leq \left\|e_i^T\, \Lambda_1\right\|\, \|c_{ik-1}\| \quad , \qquad k \geq 1 \tag{38}$$

$$\|c_{ik}\| \leq \frac{1}{\Delta\lambda_i} \left\{ \|\Lambda_1\|\, \|c_{ik-1}\| + \sum_{j=1}^{k} |\lambda_{ij}|\, \|c_{ik-j}\| \right\} \quad , \qquad k \geq 1 \quad , \tag{39}$$

where $\|\cdot\|$ denotes a compatible matrix and vector norm and,

$$\Delta\lambda_i = \min_{1 \leq j \leq n, j \neq i} |\lambda_{io} - \lambda_{jo}| \quad , \qquad 1 \leq i \leq n \quad . \tag{40}$$

Now, let $\{\bar{\lambda}_{ik}, \bar{c}_{ik}; k \geq 1\}$ denote the solution to the difference equation obtained when the inequalities in (38) and (39) are replaced by equalities, with the initial conditions

$$\bar{c}_{io} = \|c_{io}\| = \|e_i\| = 1 \quad ; \qquad \bar{\lambda}_{io} = \lambda_{io} \quad .$$

Also let

$$\bar{\lambda}_i(p) = \sum_{k=0}^{\infty} \bar{\lambda}_{ik}\, p^k \quad , \qquad \bar{c}_i(p) = \sum_{k=0}^{\infty} \bar{c}_{ik}\, p^k \quad , \tag{41}$$

whenever the series converge. Using well-known z-transform identities,

$$\bar{\lambda}_i(p) - \lambda_{io} = \left\|e_i^T\, \Lambda_1\right\|\, p\, \bar{c}_i(p) \tag{42}$$

$$\bar{c}_i(p) - \bar{c}_{io} = \frac{1}{\Delta\lambda_i} \left\{ \|\Lambda_1\|\, p\, \bar{c}_i(p) + (\bar{\lambda}_i(p) - \bar{\lambda}_{io})\, \bar{c}_i(p) \right\} \tag{43}$$

Manipulating (42) and (43), we obtain the following quadratic equation

$$\Delta\bar{\lambda}_i(p) - \left\|e_i^T\ \Lambda_1\right\|\ p = \frac{1}{\Delta\lambda_i}\left\{\|\Lambda_1\|\ p\ \Delta\bar{\lambda}_i(p) + \Delta\bar{\lambda}_i^2(p)\right\} \tag{44}$$

$$\bar{\lambda}_i(p) - \lambda_{io} = \frac{\Delta\lambda_i - \|\Lambda_1\|\, p}{2} \pm \sqrt{\left[\frac{\Delta\lambda_i - \|\Lambda_1\|\, p}{2}\right]^2 - \left\|e_i^T\ \Lambda_1\right\| p} \tag{45}$$

After some manipulation, it can be shown that

$$|\lambda_i(p) - \lambda_{io}| \leq \left|\Delta\bar{\lambda}_i(p)\right| = \left|\bar{\lambda}_i(p) - \bar{\lambda}_{io}\right| \tag{46}$$

$$|\lambda_i(p) - \lambda_{io}| \leq \frac{\Delta\lambda_i - \|\Lambda_1\, p\|}{2}\left[1 - \sqrt{1 - \frac{4\Delta\lambda_i\left\|e_i^T\,\Lambda_1\, p\right\|}{(\Delta\lambda_i - \|\Lambda_1\, p\|)^2}}\right]\ ,\ 1 \leq i \leq n \tag{47}$$

whenever

$$\alpha_i = \frac{\left\|e_i^T\ \Lambda_1\right\|}{\|\Lambda_1\|}\ ,\ \ \|\Lambda_1\, p\| \leq 2\Delta\lambda_i\left[\ \frac{1}{2} + \alpha_i - \sqrt{\alpha_i^2 + \alpha_i}\ \right]\ . \tag{48}$$

Thus, the change in each of the n-eigenvalues has to be within the bound given by (47) as the parameter p varies around its reference value of zero. It is of interest to note that near the reference operating point, (47) can be approximated by

$$|\lambda_i(p) - \lambda_{io}| \leq \frac{\left\|e_i^T\,\Lambda_1\, p\right\|}{1 - \dfrac{\|\Lambda_1\, p\|}{\Delta\lambda_i}}\quad ,\qquad 1 \leq i \leq n\quad . \tag{49}$$

It is seen that these upper bounds depend on three real variables; namely, $\Delta\lambda_i$, $\|\Lambda_1\, p\|$ and $\left\|e_i^T\,\Lambda_1\, p\right\|$. In other words, the upper bound is valid and remains unchanged for any matrix $A(p)$ for which these three variables are constant. This will be discussed in more detail in the next section. The dependence on $\Delta\lambda_i$ which is a measure of the closeness of the i^{th} eigenvalue to the remaining eigenvalues is somewhat intriguing. From (36) and (37), it is seen that when two eigenvalues are close to each other, the sensitivity of the corresponding eigenvalues (as measured by c_{ik}) to variations in the matrix is greater. The greater sensitivity in the eigenvectors then produces a greater sensitivity in the eigenvalues to variations in the matrix.

It is also useful to note a more conservative upper bound on the eigenvalue variations. From (34), it can be shown that

$$(\lambda_{io}\, I - \Lambda_o - \Lambda_1\ p)\ c_i(p) = -(\lambda_i(p) - \lambda_{io})\ c_i(p) \tag{50}$$

$$|\lambda_i(p) - \lambda_{io}| \leq \|\lambda_{io}\, I - \Lambda_o - \Lambda_1\ p\| \quad , \qquad 1 \leq i \leq n \qquad . \tag{51}$$

In many cases, upper bounds such as these are too conservative to be useful. Note, for example, that at the reference of $p = 0$, the upper bound is not zero but $\|\lambda_{io}\, I - \Lambda_o\|$ which may well be quite large. This underscores the significance of the upper bound developed in this section given by (47).

III. SOME MEASURES OF SYSTEM SENSITIVITY

In the design of a control law, achieving a robust closed-loop system is among the most important goals. By a robust system, we mean one in which the significant system characteristics are "insensitive", or have "little" sensitivity, to the expected variations in the system. For example, we may be interested in the sensitivity of location of the closed-loop system eigenvalues to variations in one or more system parameters. Thus, a measure of sensitivity would be given by the variation in one system characteristic caused by a variation in some other system property.

It is important to note that a system may be insensitive to variations in some system parameters but be highly sensitive to other system parameters. Conversely, some system characteristics may be insensitive while others are highly sensitive to the same system parameters. For example, a system may well have a high stability margin, but have a d.c. gain which varies considerably as the operating point varies. Therefore, a single number or a single measure of robustness may not always adequately describe the robustness of a system.

It is important to note that the system eigenvalues being insensitive to expected parameter variations does not necessarily imply that the location of the eigenvalues is the best for tracking commands. In other words, it is assumed that the designer has found the system's nominal response characteristics, such as the location of the system eigenvalues, noise attenuation, etc., to be adequate for the basic purpose of the system. Therefore, in this study, sensitivity analysis is intended to evaluate the extent to which the system's response characteristics vary from their nominal behavior. Sensitivity analysis does not evaluate the system's performance of its basic purpose; e.g., tracking a time-varying command, or landing an aircraft. In fact, often a high performance system will be sensitive to critical system parameters.

In most cases of interest, the system considered has parameters which are either not known with sufficient accuracy or which vary significantly over the system's operating range. Thus, it is natural to consider the system's sensitivity to variations in these parameters. However, the designer must also consider the system's sensitivity to a number of other elements such as unmodeled dynamics, nonlinear dynamic effects, transport delays, sampling rate variations, component failures, disturbances, biases, etc. While investigating all of these elements is well beyond the scope of this study, it is important to note that each of these cases can be studied using the approach developed. This is an important difference between the approach developed here and the use of σ-plots as measures of robustness. Important design questions such as the change in a critical mode or eigenvalue due to jitter in the sampling rate or other parameters cannot be readily formulated using the latter approach.

Possibly the most important of the elements mentioned above is the case of unmodeled dynamics. The way in which unmodeled dynamics affect the sensitivity of a system will be discussed next.

A. UNMODELED DYNAMICS

Most physical systems of practical interest are relatively complex nonlinear systems. To simplify their analysis, these systems are linearized at specific operating conditions within a operating range of interest and their approximations are introduced by neglecting terms of relatively small magnitudes. When the neglected terms contain variables which are themselves generated by a dynamic system, the effect of the approximation is difficult to assess and must be dealt with considerable care.

While the neglected terms are generally of second order, the closed-loop system may nevertheless have desirable characteristics which are sensitive to these admittedly small terms. To avoid high sensitivity to unmodeled dynamics, attention must be paid in the design of the control law.

Consider the system

$$\dot{x}_m = A_m\, x_m + A_{mu}\, x_u + B_m u \qquad , \tag{52}$$

$$\dot{x}_u = A_{um}\, x_m + A_u\, x_u + B_u u \tag{53}$$

where x_m and x_u are the modeled and unmodeled state vectors, respectively. The term $A_{mu}\, x_u$ is neglected, the usual justification being that the part of x_m due to the term $A_{mu}\, x_u$ is much smaller than the part due to $B_m u$. Thus, the modeled part of the system is

$$\dot{x}_m = A_m\, x_m + B_m u \quad . \tag{54}$$

This is equivalent to simply setting A_{mu} equal to zero in the "true" system given by (52) and (53), where the unmodeled state, x_u, now has no impact on the modeled state, x_m. Therefore, the eigenvalues of the modeled system are those of A_m and A_u as can be seen from the system matrix

$$A_o = \begin{pmatrix} A_m & 0 \\ A_{um} & A_u \end{pmatrix} \quad , \tag{55}$$

where A_{mu} has been set to zero. It can be shown that the matrix whose columns are the eigenvectors of A_o can be partitioned as

$$X_o = \begin{pmatrix} X_m & 0 \\ X_{um} & X_u \end{pmatrix} \quad , \quad X_o^{-1} = \begin{pmatrix} X_m^{-1} & 0 \\ X_{um}^{-1} & X_u^{-1} \end{pmatrix} \quad , \tag{56}$$

where X_m and X_u are the eigenvector matrices of A_m and A_u, respectively, and

$$X_{um}\, e_i = x_{umi} = (\lambda_{mi} I - A_u)^{-1}\, A_{um}\, X_m\, e_i \quad , \quad 1 \le i \le n_m \tag{57}$$

$$e_i^T\, X_{um}^{-1} = e_i^T\, X_u^{-1}\, A_{um}\, (\lambda_{ui} I - A_m)^{-1} \quad , \quad 1 \le i \le n_u \tag{58}$$

$$X_m^{-1} = (X_m)^{-1} \quad , \quad X_u^{-1} = (X_u)^{-1} \quad . \tag{59}$$

Now the actual system matrix A differs from the modeled system matrix A_o, by ΔA

$$\Delta A = \begin{pmatrix} 0 & A_{mu} \\ 0 & 0 \end{pmatrix} \quad , \quad A = A_o + \Delta A \quad . \tag{60}$$

Thus, for this system

$$\Delta \Lambda = X_o^{-1} \, \Delta A \, X_o = \begin{pmatrix} X_m^{-1} \, A_{mu} \, X_{um} & X_m^{-1} \, A_{mu} \, X_u \\ X_{um}^{-1} \, A_{mu} \, X_{um} & X_{um}^{-1} \, A_{mu} \, X_u \end{pmatrix} \tag{61}$$

Using (29) and interpreting p to be unity, the first order change in the "true" eigenvalues is

$$\lambda_{i1} = e_i^T \, X_m^{-1} \, h_m(\lambda_i) X_m \, e_i \quad , \quad 1 \leq i \leq n_m \tag{62}$$

$$\lambda_{i1} = e_i^T \, X_u^{-1} \, h_u(\lambda_i) X_u \, e_i \quad , \quad n_m + 1 \leq i \leq n_m + n_u \tag{63}$$

where

$$h_m(\lambda_i) = A_{mu}(\lambda_i I - A_u)^{-1} \, A_{um} \quad , \quad 1 \leq i \leq n_m \tag{64}$$

$$h_u(\lambda_i) = A_{um}(\lambda_i I - A_m)^{-1} \, A_{mu} \quad , \quad n_m + 1 \leq i \leq n_m + n_u \tag{65}$$

If the matrix A_m also varies by ΔA_m, then the combined first order change in the "true" eigenvalues would be

$$\lambda_{i1} = e_i^T \, X_m^{-1} \, [\Delta A_m + h_m(\lambda_i)] \; X_m \, e_i \quad , \quad 1 \leq i \leq n_m \tag{66}$$

From (66), it can be seen that up to first order, the effect of unmodeled dynamics on the system eigenvalues is essentially the same as that of parameter variations. When dealing with unmodeled dynamics, the transfer function, $h_m(s)$, evaluated at the i^{th} eigenvalue of the model takes the place of the system parameter variations, ΔA_m.

It is important to note that whereas ΔA_m is usually a real matrix, $h_m(\lambda_i)$ is usually a complex matrix whose elements have both magnitude and phase. So that unmodeled dynamics of the same magnitude as parameter variations may produce larger changes in the corresponding eigenvalues if their phases are appropriately aligned. Thus, the system's sensitivity to magnitude and phase variations is a measure of robustness with respect to unmodeled dynamics.

Due to the mutual coupling of the modeled and unmodeled dynamics, the eigenvalues of the unmodeled dynamics also vary as the coupling, A_{mu}, between the modeled and unmodeled, varies. This can be seen by observing (63) and (65). Therefore, if appropriate attention is not paid, the unmodeled dynamics may produce unacceptable and sometimes unstable modes.

As can be seen from (64) and (65), in order to neglect dynamic modes satisfactorily, $h_m(\lambda_i)$ and $h_u(\lambda_i)$ must be small. For the transfer functions $h_m(\lambda_i)$ and $h_u(\lambda_i)$ to be small it is necessary that the coupling terms A_{mu} and A_{um} be small, and, in particular, for the eigenvalues of A_m and A_u to be far apart from each other. The latter condition implies that the terms $\{(\lambda_i I - A_m)^{-1}, 1 \leq i \leq n_m\}$ and $\{(\lambda_i I - A_u)^{-1}, n_m + 1 \leq i \leq n_m + n_u\}$ are sufficiently small. It is important to note if the unmodeled dynamics have eigenvalues within the same range as the closed-loop modeled eigenvalues, then problems are likely to arise except for special circumstances. Thus, if the two sets of eigenvalues are not sufficiently separated, it is a safer policy not to neglect the modes under consideration in the system model. The use of stochastic optimal output feedback is usually quite convenient in such cases.

In many cases of practical interest, it is convenient to neglect the dynamics associated with the control actuators and with the sensors used for feedback. This can reduce the order, hence the complexity, of the system under consideration significantly. Also the higher order structural modes of a physical system as well as the higher harmonics of the electronic subsystem are usually neglected. These unmodeled dynamics usually consist of higher frequency modes than the modeled modes.

From the previous discussion and (62) - (66), it is clear that to obtain insensitivity to unmodeled dynamics, it is necessary that the designed closed-loop eigenvalues be separated from the unmodeled eigenvalues. Since these unmodeled eigenvalues are of high frequency and can vary significantly, it is desirable to design the closed-loop eigenvalues with sufficiently low natural frequencies so as not to excite the unmodeled eigenvalues. Therefore, unmodeled dynamic effects place an upper limit on the closed-loop eigenvalues and on the gains that can be used in the control law design. System sensitivity to other effects also place an upper limit on the closed-loop eigenvalues, as will be seen in the following section.

When considering the system sensitivity to unmodeled dynamics using the approach developed here, it is clear that the control law to be designed must not result in closed-loop eigenvalues which are "too far" into the left half-plane or too high frequency. In other words, the control law gains must not be "too high" if we are to avoid high sensitivity to unmodeled dynamics and maintain robustness. While most control designers of actual systems know the rule-of-thumb that "high-gain" control laws produce problems, usual robustness measures do not reflect the trend of high sensitivity introduced by high gain systems.

B. AN UNMODELED SENSOR DYNAMICS EXAMPLE

The meaning and usefulness of the measures of sensitivities being developed in this study would be better appreciated when applied to an example. We will select a simple, but representative, example to illustrate each of the measures investigated here.

Consider a projectile which is approximated by a point mass m. The velocity and thrust along the x-direction are $v_x(t)$ and $T_x(t)$, respectively. It is assumed that the altitude profile is independently controlled by the vertical component of the thruster, T_z. As the nominal system, we will assume that the drag on the projectile is negligible; so that the nominal system is given by

$$\dot{x} = v_x \quad , \tag{67}$$

$$\dot{v}_x = \frac{T_x}{m} \quad . \tag{68}$$

Suppose that we want to design a control law which regulates the velocity $v_x(t)$ about a commanded value v_{xc}. We would like to have no steady-state error in the presence of steady winds, and despite the fact that the mass, m, may vary $\pm$ 10% due to loss of fuel during flight and variations in the initial mass, and that the aerodynamic forces can be reduced to the drag force resulting in

$$\dot{v}_x = -a_2 \; v_x + \frac{1}{m} \; T_x + w \quad , \tag{69}$$

where the coefficient a_2 may vary within the set [0, .2] and $w(t)$ is a Gaussian white noise process with zero mean.

Let us formulate the nominal state model for this problem in the following way.

$$x_2(t) = v_x(t) \quad , \quad z(t) = v_{xc}(t) \quad , \tag{70}$$

$$b = \frac{1}{m} \quad , \quad u(t) = T_x(t) \quad , \tag{71}$$

$$\dot{x}_1 = x_2 - z \quad , \tag{72}$$

$$\dot{x}_2 = b \; u + w \quad . \tag{73}$$

Note that x_1 is the integral of the velocity error. This control structure is chosen specifically to meet the requirement that the steady-state velocity error be zero. Using the

Stochastic Optimal Feedforward/Feedback methodology, close the loop using any acceptable feedback design.

Suppose that the following controller meets the designer's closed-loop requirements.

$$u = -k_1\, x_1 - k_2\, x_2 + v \qquad , \tag{74}$$

where v contains the feedforward terms.

Closing the loop, we obtain

$$\dot{x}_1 = x_2 - z \qquad , \tag{75}$$

$$\dot{x}_2 = -b\, k_1\, x_1 - b\, k_2\, x_2 + b\, v + w \qquad . \tag{76}$$

In this section, we will investigate the implications of unmodeled dynamics on this system using the sensitivity measure developed in the preceding section. In the following sections, the same system will be investigated using the system sensitivity to other elements. Where convenient, the sensitivity measures developed here are compared to other robustness measures.

From the feedback control law shown in (74), it is clear that a measurement of the velocity, x_2, is necessary to implement the controller; x_1 can be computed as part of the control law and therefore is known.

Suppose that this velocity is being measured using an on-board sensor which may be described by

$$\dot{x}_u = -\alpha\, x_u + \alpha\, x_2 + w_u \qquad , \qquad \alpha \geq 0 \tag{77}$$

where x_u is the actual sensor output and w_u is the random sensor noise. The neglected sensor dynamics may produce effects which must be accounted for. Sensors which have relatively fast dynamics (i.e., relatively high α value) can justifiably be neglected in the design model provided that the control designer places appropriate restrictions on the control law. As will be apparent from the following, significant consequences may result from neglecting even fast sensors if the control law is not properly designed.

To investigate the effects of the unmodeled sensor using the sensitivity measure developed in the last section, first note that the actually implemented control law will not be (74), but will be

$$u = -k_1\ x_1 - k_2\ x_u + v \quad . \tag{78}$$

To put the example into the form used in the last section, we rewrite the closed-loop system with the change of variables

$$\Delta x_u = x_u - x_2 \tag{79}$$

$$\dot{x}_2 = -b\ k_1\ x_1 - b\ k_2\ x_2 - b\ k_2\ \Delta x_u + b\ v + w \tag{80}$$

$$\Delta \dot{x}_u = -\alpha\ \Delta x_u - \dot{x}_2 + w_u \tag{81}$$

$$\Delta \dot{x}_u = +b\ k_1\ x_1 + b\ k_2\ x_2 - (\alpha - b\ k_2)\ \Delta x_u - b\ v + (w_u - w) \tag{82}$$

Thus, the closed-loop system matrix for the state vector $(x_1\ x_2\ \Delta x_u)^T$ is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -bk_1 & -bk_2 & -bk_2 \\ bk_1 & bk_2 & -(\alpha - bk_2) \end{pmatrix} \tag{83}$$

$$A_u = -(\alpha - bk_2) \quad , \quad A_{mu} = \begin{pmatrix} 0 \\ -bk_2 \end{pmatrix} \quad , \quad A_{um} = (bk_1\ \ bk_2) \quad . \tag{84}$$

From (82) as well as (83) and (84), it is clear that as the gain k_2 increases, the dynamics for Δx_u in (82) becomes less stable, until it actually becomes unstable! It is seen that high gains have the tendency of making a system highly sensitive to unmodeled dynamics.

To compute the sensitivity to unmodeled sensor dynamics, we first obtain the transfer function, $h_m(\lambda_i)$, given by (64).

$$h_m(\lambda_i) = \begin{pmatrix} 0 & 0 \\ \frac{-b^2\, k_1\, k_2}{\lambda_i + \alpha - bk_2} & \frac{-b^2\, k_2^2}{\lambda_i + \alpha - bk_2} \end{pmatrix} \quad , \qquad i = 1\ ,\ 2 \quad . \tag{85}$$

The sensitivity of the system eigenvalues to the unmodeled sensor dynamics can be found to be

$$\lambda_{11} = \frac{1}{\lambda_{20} - \lambda_{10}} \quad \frac{b^2\, k_2}{\lambda_{10} + \alpha - bk_2}\,(k_1 + k_2\, \lambda_{10}) \tag{86}$$

$$\lambda_{11} = b\ \frac{\left(k_1 - \frac{bk_2^2}{2}\right) + \frac{bk_2^2}{2}\sqrt{1 - \frac{4k_1}{bk_2^2}}}{\frac{bk_2}{2}\left[1 - \frac{4k_1}{bk_2^2}\right] + \left(\alpha - \frac{3bk_2}{2}\right)\sqrt{1 - \frac{4k_1}{bk_2^2}}} \tag{87}$$

Observing (86), the basic elements that affect robustness with respect to unmodeled sensor dynamics become clear. The system sensitivity to unmodeled sensor dynamics increases as

1) the control gains k_1 and k_2 increase

2) $|\lambda_{io} + \alpha - bk_2|$ decreases

3) the distance between the eigenvalues, $|\lambda_{10} - \lambda_{20}|$ decreases.

Most control designers quickly become aware of the fact that as the feedback gains are increased, the system robustness is significantly reduced. However, if the system gains are kept low, the control performance during transients due to disturbances or noise is not satisfactory. Determining gains which are high enough for acceptable transient response and tracking, yet which are low enough to be insensitive to unmodeled dynamics is among the most difficult control design problems. The fact that the sensitivity measure developed here provides insight into the reasons for some well-known rules-of-thumb in control design makes this measure of robustness useful.

However, currently the most popular measure of robustness for multi-input multi-output systems, namely the smallest singular value of the return difference, fails to indicate that the system sensitivity to unmodeled dynamics (in fact, other elements as well) increases significantly with increasing control gains. In fact, just to the contrary, the measure of robustness seems to claim that as the feedback gains increase the robustness of the system increases. To illustrate the point, note that the smallest singular value of the return difference for the example system is

$$\underline{\sigma}\{I + G(jw)\ K(jw)\} = \underline{\sigma}\left\{1 + \frac{b(k_1 + k_2\, jw)}{(jw)^2}\right\} \tag{88}$$

$$= \left[\left(1 - \frac{bk_1}{w^2}\right)^2 + \frac{(bk_2)^2}{w^2}\right]^{1/2} \quad , \tag{89}$$

where $G(jw)$ is the open-loop transfer function from u to x_1. It is seen that as k_1 and k_2 increase the smallest singular values also increase at every frequency. This implies that the system with the higher control gains is the more robust one. Clearly, the singular value of the return difference seems to contain no hint or clue about high gains causing high sensitivity to unmodeled dynamics which may even result in instability.

It should be noted that this is not an anecdotal occurrence. Singular value analysis seems to show a preference for high gain systems in almost all situations. Let $G(s)$ be the open-loop plant transfer function and $K(s)$ the equivalent feedback control transfer function. Then, singular value analysis suggests the smallest singular value of the return difference matrix as a measure of system robustness; i.e.,

$$\underline{\sigma}\{I + G(jw)\ K(jw)\} \quad . \tag{90}$$

As the control transfer function increases in norm, eventually the smallest singular value of (90) will also increase. This implies that if the control gains are high enough, the system will be robust, which does not agree with the sensitivity results shown in this section. Therefore, is seems possible that a system be thought to be robust according to singular value analysis while having high sensitivity to unmodeled dynamics, as illustrated by the simple example in this section. Although it is possible to limit consideration to lower gain designs, the fact that the signular values of the return difference do not reflect sensitivity to unmodeled dynamics within the set limits remains unchanged.

C. EIGENVALUE AND EIGENVECTOR SENSITIVITY

In the previous sections, a sensitivity measure for the effects of unmodeled dynamics was developed and its use was demonstrated with a simple example where the sensor dynamics were neglected in the design model. In this section, we will investigate the sensitivity of the closed-loop eigenvalues to system parameter variations. In particular, we will consider the location of eigenvalues, the sensitivity of the damping ratios and the stability margin. Finally, the sensitivity measures developed will be illustrated using the example of the last section.

System parameter variations are quite common in practical systems. The question of sensitivity of eigenvalues occurs in three related but different situations. First, it is desirable to know whether a relatively small change in one or many parameters produces an inordinate change in any one eigenvalue. If these parameters can be expected to vary under the operating conditions of the plant, then it may be desirable to modify the feedback control law to reduce high sensitivities. It is further desirable to know the direction in which the eigenvalue will move to evaluate whether this sensitivity is indeed undesirable.

Second, and probably most important, it is necessary to establish that the closed-loop eigenvalues and eigenvectors remain within an acceptable range of values while the system parameters vary over the operating range. In this case, the system parameters admittedly will have significant variations. It is also to be expected that the system response, as described by its eigenvalues and eigenvectors, will have correspondingly non-negligible variations. It is important however, that the system response remain within an acceptable range. In cases where this is not possible, it becomes necessary to use gain scheduling or nonlinear designs in order to maintain the system response satisfactory throughout the operating range.

Finally, the designer must consider at least some of the more common, but accidental, large parameter variations. For example, component failures, accidental human errors, etc.; in other words, situations which are not within the normal operating modes, but which may at least temporarily occur. In such cases, it is desirable that the system maintain its stability, even if the system response is not within the satisfactory range.

Now consider an analog or digital system described by (1) or (2), respectively. Suppose that the control designer has selected the following control laws using the feedback vector, y.

$$y = C\,x + v \quad , \quad y_k = C\,x_k + v_k \quad ; \tag{91}$$

$$u = -K\,y = -K\,C\,x - K\,v + \bar{u} \;,\; u_k = -K\,y_k = -K\,C\,x_k - K\,v_k + \bar{u}_k \tag{92}$$

where v and v_k represent analog and discrete sensor noise and errors, and $\bar{u}$ and $\bar{u}_k$ are the analog and discrete feedforward controls, respectively. Closing the loop with these controllers,

$$\dot{x} = (A - B\;K\;C)\;\;x + B\;\bar{u} + w - B\;K\;v \quad , \tag{93}$$

$$x_{k+1} = (\phi - \Gamma\;K\;C)\;\;x_k + \Gamma\;\bar{u}_k + w_k - \Gamma\;K\;v_k \quad , \tag{94}$$

where w and w_k represent analog and discrete plant noise, respectively.

The feedback design may be obtained using any method which pays the necessary attention to the stochastic plant and measurement noises as well as the requirements of the feedforward controller. These include classical and modern, frequency and time domain approaches to control design. However, in this study, the method of feedback design is not relevant. Whatever the method used, here we would like to develop criteria which will help evaluate the sensitivity of the closed-loop system response to system parameter variations.

First note that, from (93) and (94), we are interested in the sensitivity of the eigenvalues and eigenvectors of the closed-loop system matrices $(A - B\,K\,C)$ or $(\phi - \Gamma\,K\,C)$ according to whether the system is continuous or discrete, respectively. Since the form of both system matrices is the same, we will treat only $(A - B\,K\,C)$ with the realization that A and B can be replaced in the formulas by ϕ and Γ, respectively, when dealing with a discrete system.

1. Small Parameter Variations.

When considering small changes in the system parameters, it is usual to use the first derivative of the eigenvalue with respect to the parameter as a measure of sensitivity. Observation of the infinite series representation in (5) reveals that the first term, λ_{i1}, is in fact the derivative of the i^{th} eigenvalue with respect to the parameter, p. From (29), it is seen that if the system matrix varies in the form of

$$A(p) = (A_o - B_o\,K\,C) \qquad + \qquad (A_1 - B_1\,K\,C) \;\; p \tag{95}$$

Then the sensitivity of the i^{th} eigenvalue, λ_{io}, to the parameter p may be defined as

$$\lambda_{i1} = e_i^T\; X_o^{-1}\; (A_1 - B_1\,K\,C)\;\; X_o\; e_i \qquad , \qquad 1 \leq i \leq n \tag{96}$$

where X_o is the eigenvector matrix of $(A_o - B_o\,K\,C)$; i.e., the nominal closed-loop system matrix.

A measure of the average sensitivity can be obtained by

$$S_p = \left[\frac{1}{n} \sum_{i=1}^{n} |\lambda_{i1}|^2 \right]^{1/2} \tag{97}$$

It should be noted that while S_p may provide a quick idea of the eigenvalue sensitivity, the information about the direction and the particular eigenvalue has been eliminated in the process.

Now, consider the $(i,\, j)$ element of the system matrix to be the parameter in question; then we can define the $(i,\, j)$ element of the Sensitivity Matrix, say S, as

$$S = \left[\frac{1}{n} \sum_{k=1}^{n} \left| X_{ki}^{-1} \; X_{jk} \right|^2 \right]^{1/2} \quad , \quad 1 \le i\,, j \le n \tag{98}$$

where X_{ij}^{-1} and X_{ij} are the (i, j) elements of X_o^{-1} and X_o, respectively.

In other words, the (i, j) element of the Sensitivity Matrix is the average change in the system eigenvalues due to a unit change in the (i, j) element of the closed-loop system matrix.

A first order approximation to the Stability Margin Matrix, M, may be obtained by

$$M_{ij} = \min_{1 \le k \le n} \left| \frac{Re\{\lambda_{ko}\}}{Re\{X_{ki}^{-1} \; X_{jk}\}} \right| \quad , \quad 1 \le i\,, j \le n \quad . \tag{99}$$

It is seen that M_{ij} is the change in the (i, j) element of the closed-loop system matrix which would produce instability, assuming the first order approximation holds. The matrix M provides relative, local information about the stability margin. Higher order approximations of the stability margin can be obtained by adding more terms such as λ_{i2} , λ_{i3}, etc. In other words, using a higher order approximation of the power series for the eigenvalue, it is possible to obtain as much accuracy about the stability margin as required.

A 4^{th} order longitudinal dynamics model for a Boeing 737 aircraft at different airspeeds is analyzed in Tables 1, 2 and 3 using the Sensitivity Matrix and the Stability Margin Matrix proposed above. In Table 1, the aircraft nominal speed is 125 knots (V_o). The parameter under consideration, p, is defined as the deviation of the airspeed from the selected nominal value. The aircraft dynamics, hence the system matrix, $A(p)$, varies with the deviation from the nominal airspeed as shown in Table 1. The Sensitivity Matrix indicates the relative impact of the system matrix elements on the eigenvalues. For example, the average change in the absolute value of the system eigenvalues to a unit change in the (2, 1) element is 0.255, while for the (3, 1) element it is 30.3 and for the (2, 4) element it is 0.00196, up to first order.

It is clear that the aircraft eigenvalues are highly sensitive to the (3, 1) element of the system matrix, whereas the sensitivity to the (2, 4) element is quite small. Therefore, the relative importance of the various elements of the system matrix is easily seen by observing the Sensitivity Matrix. On the other hand, if the airspeed varies from the nominal value by 1 knot, all the system elements vary simultaneously. In that case, the average change in the eigenvalues is given by S_p which is 0.00998 for the example in Table 1.

Two measures of stability margin are shown in Tables 1, 2 and 3. M is the margin to real parameter changes, whereas M_u is the margin to unmodeled dynamics. As shown in Section III A (Eq. 66), the first order change due to unmodeled dynamics can be obtained by replacing the real parameter variation matrix (ΔA_m) by the unmodeled transfer function matrix, $h_m(\lambda_k)$. Thus, the elements of M_u are defined as the change in $h_m(\lambda_k)$ which would produce instability, assuming a first order approximation and noting that $h_m(\lambda_k)$ is a complex-valued matrix. From Table 1, it is seen that the stability margin to unmodeled dynamics is smaller than the margin to parameter variations. Thus the relative stability margin of the system elements can be obtained by the matrices M and M_u.

2. Large Parameter Variations.

The local eigenvalue sensitivity measures described in the last section are quite valuable in evaluating the local robustness qualities of a system. On the other hand, in most design problems, it is necessary to consider a larger region about the nominal operating point and make sure that the system response remains within satisfactory bounds.

The usual way of thinking of system sensitivity, or robustness, is that it can be described by a number, a set of numbers and, in the case of singular value analysis, a function of frequency. These approaches provide useful information about the sensitivity of a system. In this section, we will introduce a new way of thinking about system sensitivity and robustness which, in our opinion, is better equipped to deal with realistic design problems.

In most control design problems, the design engineer has an operating range, i.e. the set of operating points, and a set of acceptable system response variations from a nominal acceptable response at the nominal operating point. For current purposes, we will limit the system response to characteristics which can be determined from the closed-loop system eigenvalues, such as natural frequency, damping ratio, stability, etc.

Thus, the main question is whether the system eigenvalues remain within the acceptable region while the system matrix varies within the operating range. It is important to keep in mind that not all the elements of the system matrix are subject to variation or uncertainty. Many elements of the system matrix remain completely constant throughout the operating range. Therefore, it is important to make use of such information to the extent possible.

When the parameter variations within the operating range are large, the sensitivity measures described in the last section are not applicable. However, the upper bound obtained in Section II C becomes a very useful tool both quantitatively and in providing insight on the design process. Now, let

$$\bar{A} = A - B\,K\,C \qquad , \qquad \bar{A}_o = A_o - B_o\,K\,C_o \tag{100}$$

$$\Delta\bar{A} = \bar{A} - \bar{A}_o = \Delta A - \Delta\,B\,K\,C - B_o\,K\,\Delta\,C \tag{101}$$

$$\Delta\Lambda = X_o^{-1}\,\Delta\bar{A}\,X_o \tag{102}$$

$$\alpha_i = \frac{||e_i^T \, \Delta \Lambda||}{||\Delta \Lambda||} \quad , \quad \beta_i = \frac{||\Delta \Lambda||}{\Delta \lambda_i} \quad , \quad x_i = \frac{\Delta \bar{\lambda}_i}{\Delta \lambda_i} \quad , \quad 1 \leq i \leq n \tag{103}$$

Then the upper bound in (47) can be rewritten as

$$\left| \lambda_i - \lambda_{io} \right| \leq \frac{1 - \beta_i}{2} \left[1 - \sqrt{1 - \frac{4 \alpha_i \beta_i}{(1 - \beta_i)^2}} \;\; \right] \Delta \lambda_i \quad , \quad 1 \leq i \leq n \tag{104}$$

$$0 \leq \alpha_i \leq \frac{(1 - \beta_i)^2}{4 \beta_i} \quad , \quad 0 \leq \beta_i \leq 1 \quad , \quad \alpha_i \leq 1 \quad , \quad 1 \leq i \leq n \tag{105}$$

Thus, given the amount of variation in the open-loop parameters, namely $\Delta A, \Delta B$ and ΔC, (104) provides a limit on the change in every eigenvalue when the three parameters $\alpha_i, \beta_i, \Delta \lambda_i$ are known. It is important to have individual limits for each eigenvalue, because large and complex systems tend to have eigenvalues at widely varying locations of the complex plane.

A few remarks are worthy of note. Possibly, the most significant point is that the change in eigenvalues is still monotonic with $\Delta \Lambda$ even though we are no longer dealing with the first order variations, but rather the total change as bounded by (104).

$$||\Delta \, \Lambda|| = ||X_o^{-1} \, (\bar{A} - \bar{A}_o) \, X_o|| = ||X_o^{-1} \, (\Delta A - \Delta \, B \, K \, C - B_o \, K \, \Delta \, C) \, X_o|| \tag{106}$$

It should be noted that whereas the system response is largely dependent on the location of the eigenvalues, the system sensitivity is highly dependent on the eigenvectors; i.e., the columns of X_o. It follows that the system eigenvectors should be selected so as to minimize the effects of the largest variations in the system parameters; i.e., $\Delta A, \Delta B$ and ΔC.

From (103) and (104), it is seen that the relative location of the eigenvalues also plays a role in the sensitivity of the eigenvalues due to $\Delta\lambda_i$, or the minimum distance between the i^{th} eigenvalue and the remaining ones. This dependence appears to be not as strong.

If the eigenvalues and eigenvectors of the system matrix can be selected independently, it would seem that the eigenvalues would be selected to produce the necessary noise attenuation, transient response, feedforward bandwidth requirements, etc. Then the eigenvectors could be selected to minimize the system's sensitivity to parameter variations resulting in $X_o \Lambda_o X_o^{-1}$ as the closed-loop system matrix $\bar{A}_o$.

However, in most systems of practical relevance, the eigenvalues and eigenvectors cannot be selected independently. If the feedback gains are selected to place the eigenvalues, usually the eigenvectors are also fixed, and the system may have very high sensitivity. Thus, pole placement alone does not usually result in acceptable robustness. Since plant and sensor noise as well as other considerations must also be included, the use of stochastic optimal output feedback multi-configuration control techniques are useful in trying to achieve an acceptable compromise among the various goals of feedback compensation for a given task.

The extent of variation or uncertainty among system parameters can vary considerably. While some parameters representing kinematic relationships do not vary at all, others may double. Therefore, it is desirable that our measure of sensitivity be able to accommodate such important information about the structure of variations. A system which is sensitive to parameters with low or no variability, but which is insensitive to parameters with high variability should be considered robust. Since the elements of $\Delta A, \Delta B$ and ΔC which do not vary can be set to zero in computing $||\Delta \Lambda||, \alpha_i$ and β_i, the structural information about the extent variability is automatically included in the upper bound of the eigenvalue change. When such structural variability is not specifically taken into account, the full condition number of the closed-loop system matrix $\bar{A}_o$ would determine the sensitivity.

$$||\Delta\,\Lambda|| \leq \left(||X_o^{-1}||\ ||X_o||\right)\ ||\bar{A} - \bar{A}_o|| \quad . \tag{107}$$

Another significant property about the sensitivity measure in (104) is that it provides upper bounds for each eigenvalue. For example, suppose the closed-loop system has a pole close to the origin, while the remaining eigenvalues are farther in the left-half-plane. The mere fact that a pole is near the origin does not, of itself, imply that the system has low stability margin. What is important is the sensitivity of the pole to parameter variations. Often the poles far in the left-half-plane are the first to become unstable.

From (104), it can be shown that if the i^{th} row of X_0^{-1} (i.e., the i^{th} row eigenvector of $\bar{A}_o$ is orthogonal to the change in the system matrix, $\Delta\bar{A}$, then the i^{th} eigenvalue remains unchanged. In fact, if α_i is small, then the total change in the i^{th} eigenvalue will also be small. In other words, if some eigenvalues need to be more robust than others, then this insensitivity may be obtained by making the corresponding row eigenvector nearly orthogonal to the system variations; i.e., $\alpha_i \approx 0$.

It should be noted that the norms in (103) for α_i and β_i can be arbitrarily selected matrix norms. For example, if the 2-norm is used, then $||\Delta\,\Lambda||$ is the largest singular value of $\Delta\,\Lambda$. Other norms such as the 1-or the ∞-norms may sometimes be more convenient.

For a fixed β_i in (104) the change in the eigenvalue $|\lambda_i - \lambda_{io}|$ is monotomic with α_i. Therefore, if α_i is set equal to 1, the inequality in (104) still holds.

It is sometimes more convenient to use the normalized variable x_i instead of the upper bound of $|\lambda_i - \lambda_{io}|$. Making this substitution and after considerable manipulation it can be found that

$$\beta_i = \frac{x_i\,(1 - x_i)}{(\alpha_i + x_i)} \quad , \qquad 1 \leq i \leq n \quad . \tag{108}$$

This equation is equivalent to (104) where x_i is the upper bound on the change in the i^{th} eigenvalue normalized by $\Delta\lambda_i$, except that it solves for β_i in terms of x_i and α_i. Note that the normalization has eliminated the variable $\Delta\lambda_i$; so that it is no longer explicitly shown in the equation. Figure 1 shows the class of plots of β_i versusx_i at various levels of α_i.

It is also possible to ask the reverse question: If it is required that eigenvalues of the system remain within given regions, what is the region of operating points, or the set of system parameters, which can be allowed? i.e., what is the operating regime?

A straight-forward approach to answer this important question may be developed as follows. Suppose the required regions for the eigenvalues can be expressed by

$$|\lambda_i - \lambda_{io}| \leq q_i \quad , \quad 1 \leq i \leq n \tag{109}$$

Then, it can be shown that any $\Delta\bar{A}$ such that

$$||X_o^{-1}\,\Delta\bar{A}\,X_o|| \leq \min_{1\leq i\leq n} \left\{ \frac{(1-x_i)}{(\alpha_i + x_i)}\; q_i \right\} \quad , \tag{110}$$

and

$$||e_i^T\,X_o^{-1}\,\Delta\bar{A}\,X_o|| \leq \alpha_i\; ||X_o^{-1}\,\Delta\bar{A}\,X_o|| \quad , \quad 1 \leq i \leq n \tag{111}$$

is in the acceptable operating range, in the sense that the system eigenvalues will satisfy (109). The selection of α_i is not always straight forward. In such cases $\alpha_i = 1$ may be used, although this is likely to produce conservative results.

Thus, with the procedure presented, it is possible to see if a variation in the system parameters produces an acceptable variation in the each eigenvalue. Or conversely, given an acceptable range of variation for each eigenvalue, a set of parameters within which the system eigenvalues are each within their acceptable range can be obtained. These procedures will be illustrated using the example system introduced in the last section.

Finally, it is seen that while only the diagonal elements of the matrix $\Delta\Lambda$ are needed to obtain the derivatives of the eigenvalues, the sensitivity to larger parameter variations is influenced by all the elements of $\Delta\Lambda$. Furthermore, the variations in the eigenvectors are also highly influenced by all the elements of $\Delta\Lambda$. Finally, the sensitivity of the system transfer function is also directly proportional to $\Delta\Lambda$, but is beyond the scope of this investigation.

Therefore, the sensitivity of a system cannot be fully understood or evaluated by only the diagonal elements of $\Delta\Lambda$, but must consider the complete matrix. From Figure 1, it is clear that the eigenvalue sensitivity and the norm $\|\Delta\Lambda\|$ are monotomic. The slope of the relationship is determined by α_i which contains information about the particular row of $\Delta\Lambda$. Therefore, when only the relative sensitivity of two systems is needed, as in a design optimization, $\|e_i^T\Delta\Lambda\|$ and $\|\Delta\Lambda\|$ provide a relative sensitivity measure that can be useful. In particular, recall the approximation

$$|\lambda_i - \lambda_{io}| \leq \frac{\|e_i^T\Delta\Lambda\|}{1 - \frac{\|\Delta\Lambda\|}{\Delta\lambda_i}} \quad , \quad 1 \leq i \leq n \quad .$$

Now, we return to the example introduced in Section III B to illustrate the use of the sensitivity analysis procedures developed in this section. The open-loop system is given by (72), (73), the control by (74) and the closed-loop system by (75), (76). The closed-loop matrices of interest can be found to be

$$A_o = \begin{pmatrix} 0 & 1 \\ -b_o k_1 & -b_o k_2 \end{pmatrix} \quad , \quad \Delta A = \begin{pmatrix} 0 & 0 \\ -(\frac{\Delta b}{b_o})b_o k_1 & -\Delta a_2 - (\frac{\Delta b}{b_o})b_o k_2 \end{pmatrix} \tag{112}$$

$$X_o = \begin{pmatrix} 1 & 1 \\ \lambda_{10} & \lambda_{20} \end{pmatrix} \quad , \quad X_o^{-1} = \frac{1}{\lambda_{20} - \lambda_{10}} \begin{pmatrix} \lambda_{20} & -1 \\ -\lambda_{10} & 1 \end{pmatrix} \tag{113}$$

$$\begin{aligned} \Delta\Lambda &= X_o^{-1}\Delta A X_o \\ &= \frac{1}{\lambda_{20} - \lambda_{10}} \begin{pmatrix} -\Delta b(k_1 + k_2\lambda_{10}) - \Delta a_2\lambda_{10} & -\Delta b(k_1 + k_2\lambda_{20}) - \Delta a_2\lambda_{20} \\ \Delta b(k_1 + k_2\lambda_{10}) + \Delta a_2\lambda_{10} & \Delta b(k_1 + k_2\lambda_{20}) + \Delta a_2\lambda_{20} \end{pmatrix} \end{aligned} \tag{114}$$

After some manipulation,

$$\Delta\Lambda = \frac{1}{\lambda_{10} - \lambda_{20}} \begin{pmatrix} \lambda_{10}^2 \left(\frac{\Delta b}{b_o}\right) - \lambda_{10}\Delta a_2 & \lambda_{20}^2 \left(\frac{\Delta b}{b_o}\right) - \lambda_{20}\Delta a_2 \\ -\lambda_{10}^2 \left(\frac{\Delta b}{b_o}\right) - \lambda_{10}\Delta a_2 & -\lambda_{20}^2 \left(\frac{\Delta b}{b_o}\right) + \lambda_{20}\Delta a_2 \end{pmatrix} \tag{115}$$

where $\lambda_{10}, \lambda_{20}$ are the nominal closed-loop system eigenvalues.

As mentioned in Section III B, the drag term $\Delta a_2\, x_2$ and the mass vary within limits which can be approximated by

$$-.2 \le \Delta a_2 \le 0 \quad , \quad \left|\frac{\Delta b}{b_o}\right| \le .1 \tag{116}$$

The designer of the feedback controller selects the gains k_1 and k_2 based on a number of objectives imposed by the task at hand such as the transient response requirements, measurement and plan noise suppression, the bandwidth of the class of feedforward commands, sensitivity to system parameters and unmodeled dynamics, etc. Once the selection of the feedback gains is made, it is of interest to check whether the eigenvalues of the closed-loop system remain within acceptable bounds for the other control objectives as the parameters Δa_2 and Δb take on various values. The procedure developed in this section may be used for this purpose.

Now, suppose that, using the nominal system, the designer has selected gains

$$b_o\, k_1 = 2 \quad , \quad b_o\, k_2 = 2 \tag{117}$$

which correspond to the critically damped eigenvalues

$$\lambda_{10} = -1 + j \quad , \quad \lambda_{20} = -1 - j \quad . \tag{118}$$

Figure 2 shows the closed-loop eigenvalues and the regions which the designer considers acceptable in terms of damping, noise suppression, etc. First compute the norm $||\Delta\Lambda||$, α_i and β_i. In this example we will use the 2-norm. After some computation,

$$\beta_i = \frac{1}{\Delta\lambda_i} \, ||\Delta A|| = \frac{\sqrt{2}}{|\Delta\lambda|^2} \left[\left| \lambda_{10}^2 \left(\frac{\Delta b}{b_o}\right) - \lambda_{10} \, \Delta a_2 \right|^2 + \left| -\lambda_{20}^2 \left(\frac{\Delta b}{b_o}\right) + \lambda_{20} \, \Delta a_2 \right|^2 \right]^{1/2} \quad i = 1, 2 \tag{119}$$

$$\Delta\lambda_i = |\Delta\lambda| = |\lambda_{10} - \lambda_{20}| = |\lambda_{20} - \lambda_{10}| = 2 \, |\lambda_{10}| \, sin\theta = 2 \quad , \quad 1 = 1, 2 \tag{120}$$

First consider Δa_2.

$$\beta_i = \frac{2|\lambda_{10}|}{|\Delta\lambda|^2} \, |\Delta a_2| = \frac{|\Delta a_2|}{sin\theta \, |\Delta\lambda|} = \frac{|\Delta a_2|}{\sqrt{2}} \tag{121}$$

$$\beta_i \leq 0.1414 \quad , \quad \alpha_i = \sqrt{2}/2 \tag{122}$$

Using (104), we find that

$$|\lambda_i - \lambda_{io}| \leq 0.278 \quad , \quad i = 1, 2 \quad . \tag{123}$$

The eigenvalues are seen to remain within an acceptable region as Δa_2 moves within its expected limits. Clearly, using a nominal design system with an a_2 value of 0.1 would reduce the change Δa_2 to [-.1, .1] and would have further margin. Also note that the value obtained is a conservative upper bound. On the other hand, the first order variation

$$\lambda_{11} \, \Delta a_2 = - \left(\frac{1 + j}{2}\right) \, \Delta a_2 \tag{124}$$

also provides important information about the eigenvalue changes, but does not contain information about the eigenvector variations. In many cases, λ_{i1} and λ_{i2} may provide sufficient accuracy. On the other hand, change in b produces a simultaneous variation in both system parameters. From (119),

$$\beta_i = \frac{2|\lambda_{10}|^2}{|\Delta\lambda|^2}\left|\frac{\Delta b}{b_o}\right| = \frac{\left|\frac{\Delta b}{b_o}\right|}{2\,sin^2\theta} = \left|\frac{\Delta b}{b_o}\right| \quad , \quad i = 1, 2 \quad . \tag{125}$$

Using (104),

$$|\lambda_i - \lambda_{io}| \leq 0.174 \quad , \quad i = 1, 2 \quad , \tag{126}$$

which is also within the acceptable region. When both Δa_2 and Δb vary, the upper bound will exceed the acceptable region. However, the region obtained is still not that large considering the conservative nature of the upper bound. A different design may be necessary if the damping requirements must be maintained.

On the other hand, it is possible to characterize the set of parameters which guarantee the eigenvalues remain within their acceptable regions.

Using (108) with α equal to $\sqrt{2}/2$, and substituting into (119) and manipulating

$$\frac{1}{2}\left|\lambda_{10}^2\left(\frac{\Delta b}{b_o}\right) - \lambda_{10}\,\Delta a_2\right| \leq .204 \tag{127}$$

Further manipulating,

$$\left(\frac{\Delta b}{b_o}\right)^2 + \left(\left(\frac{\Delta b}{b_o}\right) + \Delta a_2\right)^2 \leq .0832 \tag{128}$$

From (128), it can be seen that an operating where

$$\left|\frac{\Delta b}{b_o}\right| \leq .1 \qquad \text{and} \qquad .17 \leq \Delta a_2 \leq .17 \tag{129}$$

can be guaranteed to remain within the acceptable region of eigenvalues. However, if $\Delta a_2 = 0$, then b may vary by over $\pm$ 20%, while at a fixed b_o, Δa_2 may vary within [-.288, .288].

Finally, from (119), (121) and (125), it may be noted that insensitivity of eigenvalues and eigenvectors with respect to system parameters does not lead towards the choice of high control gains, whereas high gains can lead to high sensitivity to unmodeled dynamics. Therefore, the smallest singular value of the return difference, as shown in (89), seems to be more a measure plant noise suppression and fast transient response characteristics, rather than the sensitivity of system characteristics to plant parameters.

IV. CONCLUSIONS

In the design of the feedback control law, the closed-loop system's robustness is among the most important considerations. By robustness, we mean that the system characteristics are sufficiently insensitive to expected system parameter variations to continue to perform the basic system objective in an acceptable manner. The expected system parameters considered here are the variations of the open-loop system matrices (A, B, C) over the desired operating range, uncertainties in these parameters and neglected unmodeled system dynamics.

In this report, some measures of eigenvalue and eigenvector sensitivity which are applicable to both digital and analog systems are developed and investigated. Since the eigenvalue and eigenvectors of the closed-loop system completely specify the feedback characteristics of the closed-loop system, if the eigenstructure of the system remains within acceptable limits defined by the feedback designer, then the essential system characteristics also remain within acceptable limits, resulting in a robust system.

Formulating the system matrix as a function of a parameter, p, the eigenvalues and eigenvectors are expressed in a power series representation. The coefficients of both power series can be obtained from a recursive difference equation. Therefore, the variations in the eigenvalue and eigenvectors can be obtained as a function of p with as much accuracy as desired. Finally, an upper bound on the change in each eigenvalue and eigenvector is obtained. Due to the development of these infinite series representations, it is no longer necessary to constrain the sensitivity analysis only to first derivative information. A more complete exploration of the implications of the power series representation is left for future research.

A new sensitivity measure is developed by considering the effects of neglected unmodeled dynamics. Since almost any physical system is modeled neglecting some of the dynamics, this measure of sensitivity is useful in most applications. This sensitivity measure indicates that neglecting dynamics affects the actual system eigenvalue variations in direct proportion to its transfer function evaluated at the modeled eigenvalues. Thus, the closer the modeled and unmodeled eigenvalues get, the larger the sensitivity, and conversely. Since most unmodeled dynamics are high frequency effects, this sensitivity places a limit on the magnitude of the gains that can be used in the feedback law.

This property is demonstrated by considering a simple system design where the dynamics of a velocity sensor is neglected in the design model. It is demonstrated that a high-gain feedback results in high eigenvalue sensitivity. This demonstrates that high gains cause robustness problems due to unmodeled dynamics rather than just high control activity.

Analyzing the same example with the smallest singular value of the return difference, it is seen that this singular value is not a reliable measure of eigenvalue and eigenvector sensitivity, but may be more appropriate as a measure of plant noise suppression and fast transient response.

Finally, a measure of sensitivity to system parameter variations is developed using the upper bound previously obtained. This is used to establish the robustness of a system by showing that the expected system parameter variations produce eigenvalue variations within acceptable regions. The sensitivity measure also increases with high feedback gains although not to the same extent as sensitivity unmodeled high frequency dynamics.

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TABLE 1

EIGENVALUE SENSIIVITY (S) AND STABILITY MARGIN MATRICES FOR LONGITUDINAL DYNAMICS OF A SMALL JET AIRCRAFT

ANALYSIS PARAMETER: AIRSPEED V IN KNOTS

$$A(p) = A_o + A_1 p \quad , \quad p = (V - V_o)$$

NOMINAL FLIGHT CONDITION FOR A_o

$V_o = 125$ knots , $\phi = 0.$, $Wt = 80,000.$ lbs. , $c.g._o = .2$

$$\lambda_o = -.632 \pm j\,1.15\ ;\ -.0141 \pm j\,.16$$

A_o

u	w	q	θ
-.409E-01	.965E-01	-.540E+01	-.322E+02
-.286E+00	-.728E+00	.216E+03	-.806E+00
-.521E-03	-.621E-02	-.522E+00	.280E-15
.000E+00	.000E+00	.100E+01	.000E+00

A_1

u	w	q	θ
.573E-03	.290E-02	-.690E+00	.331E-02
.755E-04	.153E-01	-.175E+01	-.111E+00
.187E-04	.602E-04	.416E-02	-.196E-15
.000E+00	.000E+00	.000E+00	.000E+00

Sensitivity

S

u	w	q	θ
.353E+00	.976E-01	.524E-03	.186E-02
.255E+00	.357E+00	.191E-02	.196E-02
.303E+02	.654E+02	.351E+00	.309E+00
.705E+02	.111E+02	.598E-01	.365E+00

$S_p = .998\,E-2$

$M_p = 64.1$

Stability Margin

M

u	w	q	θ
.283E-01	.184E+00	.340E+02	.311E+02
.776E+00	.126E+01	.194E+03	.796E+01
.272E-01	.542E-01	.128E+01	.664E-01
.114E+01	.178E-01	.881E+00	.284E-01

M_u

u	w	q	θ
.282E-01	.181E+00	.338E+02	.544E+01
.390E-01	.251E+00	.469E+02	.753E+01
.329E-03	.212E-02	.396E+00	.636E-01
.141E-03	.909E-03	.170E+00	.272E-01

TABLE 2

EIGENVALUE SENSIIVITY (S) AND STABILITY MARGIN MATRICES FOR LONGITUDINAL DYNAMICS OF A SMALL JET AIRCRAFT

ANALYSIS PARAMETER: AIRSPEED V IN KNOTS

$$A(p) = A_o + A_1 p \quad , \quad p = (V - V_o)$$

NOMINAL FLIGHT CONDITION FOR A_o

$V_o = 135$ knots , $\phi = 0.$, $Wt = 80,000.$ lbs. , $c.g._o = .2$

$$\lambda_o = -.777 \pm j\,1.25 \; ; \; -.0161 \pm j\,.137$$

A_o

u	w	q	θ
-.466E-01	.687E-01	.106E+01	-.322E+02
-.288E+00	-.976E+00	.233E+03	.146E+00
-.706E-03	-.689E-02	-.564E+00	-.560E-15
.000E+00	.000E+00	.100E+01	.000E+00

A_1

u	w	q	θ
.573E-03	.290E-02	-.690E+00	.331E-02
.755E-04	.153E-01	-.175E+01	-.111E+00
.187E-04	.602E-04	.416E-02	-.196E-15
.000E+00	.000E+00	.000E+00	.000E+00

S Sensitivity

u	w	q	θ
.352E+00	.906E-01	.452E-03	.159E-02
.265E+00	.358E+00	.194E-02	.176E-02
.377E+02	.653E+02	.355E+00	.293E+00
.819E+02	.126E+02	.513E-01	.366E+00

$S_p = .979$ E-2

$M_p = 72.2$

M Stability Margin

u	w	q	θ
.325E-01	.213E+00	.525E+02	.423E+02
.543E+00	.156E+01	.284E+03	.103E+02
.982E-02	.364E+00	.158E+01	.715E-01
.127E+01	.355E-01	.134E+01	.324E-01

M_u

u	w	q	θ
.323E-01	.211E+00	.524E+02	.724E+01
.431E-01	.282E+00	.698E+02	.966E+01
.303E-03	.198E-02	.491E+00	.680E-01
.139E-03.	.907E-03	.225E+00	.311E-01

TABLE 3

EIGENVALUE SENSIIVITY (S) AND STABILITY MARGIN MATRICES FOR LONGITUDINAL DYNAMICS OF A SMALL JET AIRCRAFT

ANALYSIS PARAMETER: AIRSPEED V IN KNOTS

$$A(p) = A_o + A_1 p \quad , \quad p = (V - V_o)$$

NOMINAL FLIGHT CONDITION FOR A_o

$V_o = 115$ knots , $\phi = 0.$, $Wt = 80,000.$ lbs. , $c.g._o = .2$

$$\lambda_o = -.581 \pm j\,1.06 \; ; \; -.0125 \pm j\,.174$$

A_o

u	w	q	θ
-.352E-01	.127E+00	-.127E+02	-.321E+02
-.286E+00	-.670E+00	.198E+03	-.207E+01
-.332E-03	-.568E-02	-.481E+00	-.448E-14
.000E+00	.000E+00	.100E+01	.000E+00

A_1

u	w	q	θ
.573E-03	.290E-02	-.690E+00	.331E-02
.755E-04	.153E-01	-.175E+01	-.111E+00
.187E-04	.602E-04	.416E-02	-.196E-15
.000E+00	.000E+00	.000E+00	.000E+00

Sensitivity

S

u	w	q	θ
.355E+00	.101E+00	.579E-03	.204E-02
.253E+00	.359E+00	.192E-02	.213E-02
.304E+02	.653E+02	.349E+00	.335E+00
.649E+02	.978E+01	.656E-01	.366E+00

$S_p = .996$ E-2

$M_p = 58.7$

Stability Margin

M

u	w	q	θ
.252E-01	.176E+00	.255E+02	.247E+02
.302E+01	.115E+01	.151E+03	.653E+01
.207E-01	.318E-01	.119E+01	.545E-01
.777E+00	.121E-01	.620E+00	.254E-01

M_u

u	w	q	θ
.250E-01	.173E+00	.254E+02	.443E+01
.350E-01	.242E+00	.356E+02	.622E+01
.292E-03	.202E-02	.297E+00	.519E-01
.137E-03	.944E-03	.139E+00	.243E-01

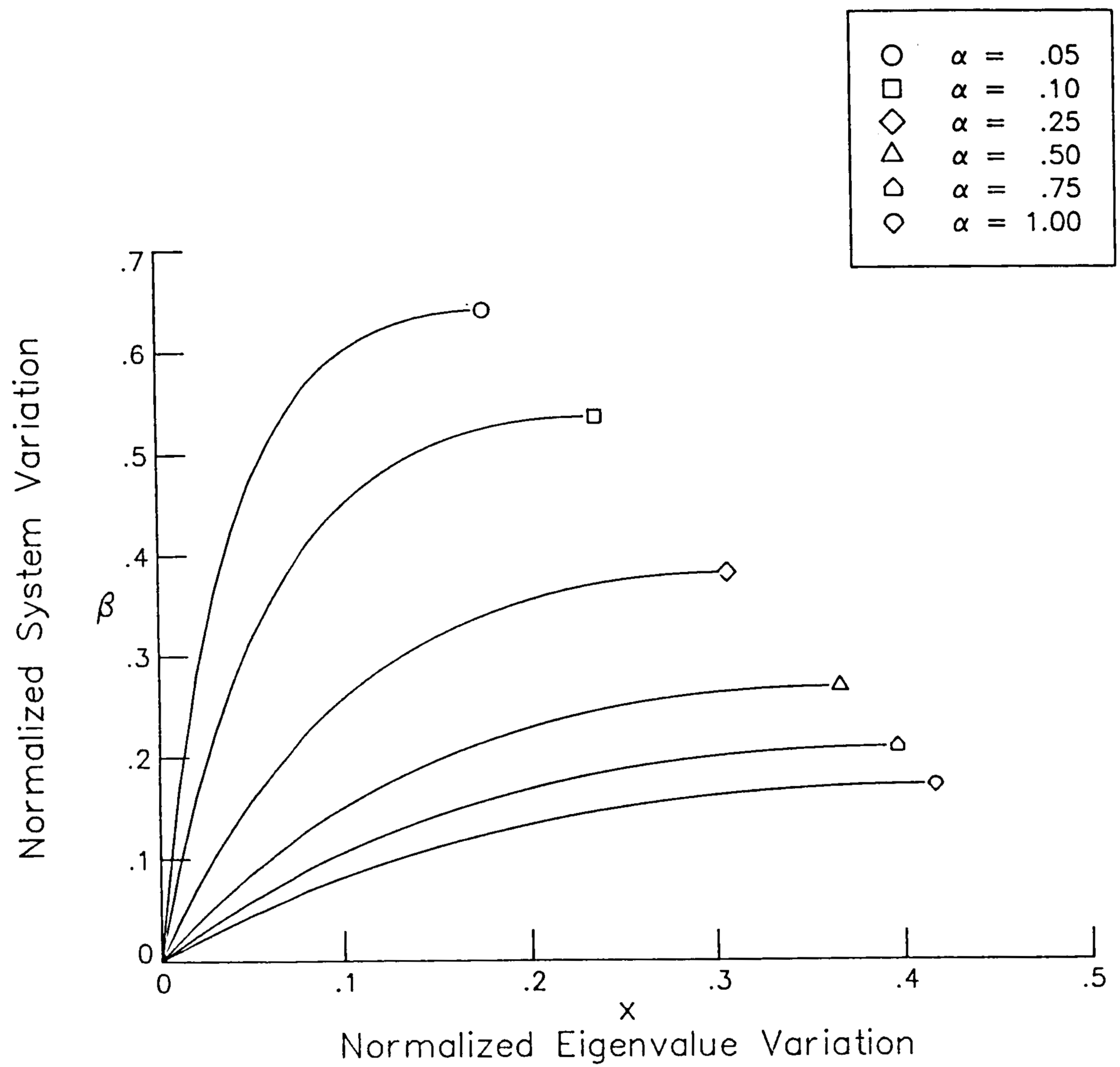

FIGURE 1. UPPER BOUNDS ON THE VARIATION OF NORMALIZED SYSTEM EIGENVALUES

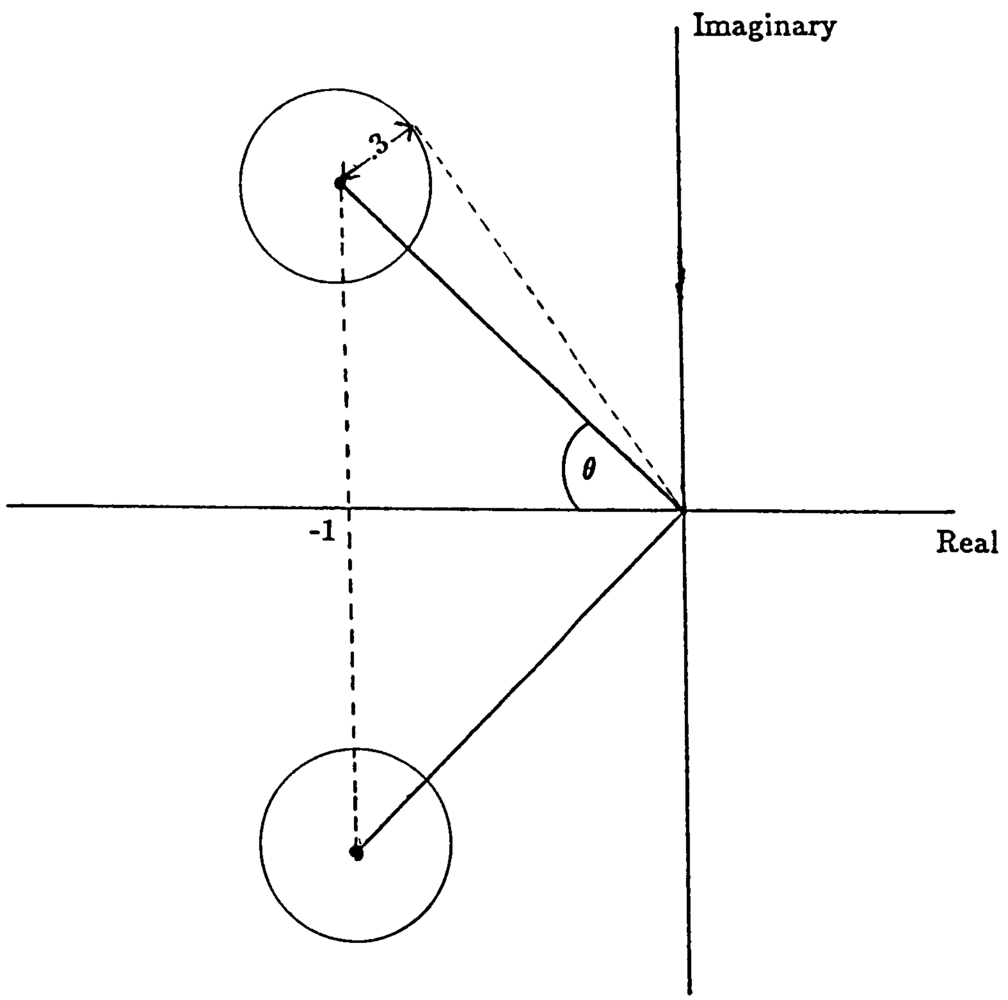

FIGURE 2. ACCEPTABLE EIGENVALUE REGIONS

NASA National Aeronautics and Space Administration

Report Documentation Page

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-4108		

4. Title and Subtitle	5. Report Date
Investigation, Development, and Application of Optimal Output Feedback Theory. Vol. IV: Measures of Eigenvalue/Eigenvector Sensitivity to System Parameters and Unmodeled Dynamics	December 1987
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
Nesim Halyo	FR-687103
	10. Work Unit No.
	505-66-41-04
9. Performing Organization Name and Address	11. Contract or Grant No.
Information & Control Systems, Inc. 28 Research Drive Hampton, VA 23666	NAS1-15759
	13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address	Contractor Report March 1985 to June 1987
National Aeronautics & Space Administration Langley Research Center Hampton, VA 23665-5225	14. Sponsoring Agency Code

15. Supplementary Notes

NASA Langley Technical Monitor: Richard M. Hueschen
(Final Report)

16. Abstract

In this report, some measures of eigenvalue and eigenvector sensitivity applicable to both continuous and discrete linear systems are developed and investigated. An infinite series representation is developed for the eigenvalues and eigenvectors of a system. The coefficients of the series are coupled, but can be obtained recursively using a nonlinear coupled vector difference equation.

A new sensitivity measure is developed by considering the effects of unmodeled dynamics. It is shown that the sensitivity is high when any unmodeled eigenvalue is near a modeled eigenvalue. Using a simple example where the sensor dynamics have been neglected, it is shown that high feedback gains produce high eigenvalue/eigenvector sensitivity. The smallest singular value of the return difference is shown not to reflect eigenvalue sensitivity since it increases with the feedback gains.

Using an upper bound obtained from the infinite series, a procedure to evaluate whether the sensitivity to parameter variations is within given acceptable bounds is developed and demonstrated by an example.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Sensitivity, Robustness, Eigenvalue, Eigenvector, Unmodeled Dynamics, Parameter Variations	Unclassified - Unlimited Subject Category 63

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	58	A04

NASA FORM 1626 OCT 86 NASA-Langley, 1987

For sale by the National Technical Information Service, Springfield, Virginia 22161-2171